ROTATIONAL COMPONENT OF AN ELASTIC WAVE FIELD IN THE NEAR ZONE OF A POINT SOURCE ON THE SURFACE

V. A. Kozlov, V. M. Matveev,
N. V. Pet'kin, and D. N. Romashko

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Rotational oscillations, as opposed to the translational oscillations usually studied, are characterized by a periodic change of coordinate axes, which are coupled to a specified volume of elastic material. Techniques have been devised for measuring such oscillations [1-3]. In particular, one [3] describes a seismic sensor for twisting oscillations, which is based on the principle of molecular electronics.

The rotation angle ψ of an elementary volume of a continuous medium is related to the deformation field U by [4]

$$\psi = (1/2)\mathrm{rot}\mathbf{U}. \tag{0.1}$$

The quantity $\ddot{\psi}$ corresponds to the angular acceleration, which can be measured directly only in a limited number of cases, when distortions from the field U inside the sensor can be neglected. The region where this expression can be applied is evaluated by modeling the motion of a spherical inclusion caused by incoming plane compression and shear waves [5, 6]. The translation and rotation of a rigid sphere with an eccentric center of mass which is embedded in a homogeneous elastic material is examined in [6]. The presence of an eccentricity means that longitudinal as well as transverse waves cause the sphere to rotate. These results show that Eq. (0.1) can be applied with satisfactory accuracy only for a well-balanced sphere, which cannot always be attained in practice. Actually, [6] contains an error in intermediate terms, and the applicability of Eq. (0.1) is much wider, as will be shown below.

The gradient character of the relationship between ψ and U is most important when $\ddot{\psi}$ detectors are used in the near zone of a source. In this case the relative contributions to the measured quanity from the closest oscillation sources grow significantly compared to those from far sources. In the near zone of a monopole source, when the deformation field has the form $U \sim |\mathbf{R}|^{-1}\exp(ik|\mathbf{R}|)$, $\mathbf{R} = (x, y, z)$, the derivative of the exponential multiplier $|\mathbf{R}|^{-1}$ is more significant than the derivative of the oscillating exponential in the spatial derivatives used to compute the cross product. Therefore the amplitude of the angular oscillations ψ depends more strongly on $|\mathbf{R}|$ by one order of magnitude $(\sim|\mathbf{R}|^{-2})$, compared with the amplitude U of the field itself. This emphasis on the contribution of near sources can be of practical interest, for example, in seismic observation systems. Here we examine the near field of a surface oscillation source in the specific but generally interesting case when the source is represented by a concentrated external force, normal to the surface and harmonic in time (Lamb problem).

Variants of this problem have been examined [7-9] for a homogeneous elastic half-space. In the case of harmonic point source the solution is computed by integration. The asymptotic field for the far zone is also presented in [8]. The near zone has practically not been examined quantitatively at all [8]. A numerical analysis of the nonstationary Lamb problem for the near-zone elastic field was attempted in [9]. However, time-dependent deformation results were presented only for one point of the spatial coordinates, which makes it impossible to judge either the structure of the field as a whole or the derivatives of its components. These quantities are required to compute the cross product of the field. Moreover, the ratio of the delay time and the characteristic length of the computed signal shows that the results are more related to the transition zone than to the near zone. Several experimental results are published in [10].

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1. We examine an elastic homogeneous half-space with an oscillation source at the surface at the origin (the Oz axis is pointed into the elastic material, normal to the surface). The source is given by a time-harmonic external point force $\mathbf{F}(t)$, on the surface: $\mathbf{F}(t) = \mathbf{F}\cdot\exp(-i\omega t)$. It the field $\mathbf{U}$ is evaluated in the near zone, where the magnitude of the deformation is caused by the stationary ($\omega = 0$) force $\mathbf{F}$ [7], then the component of the force $\mathbf{F}$ tangent to the surface often can be neglected in practice, because the relative magnitude of this component in the total field is proportional to the parameter $\gamma^2 = c_s^2/c_p^2$, where c_s and c_p are the velocities of the transverse and longitudinal waves. Therefore further calculations will be restricted to a force perpendicular to the boundary of the half-space: $\mathbf{F} = (0, 0, F_z)$. The initial system of equations and the exact integral solution are given in [8]. The Results of [8] yield an equation for $\boldsymbol{\psi}$:

$$\boldsymbol{\psi} = \mathbf{e}_\varphi \frac{F_z}{2\pi\mu}\frac{\omega^2}{c_s^2}\int_0^\infty \frac{ik_z k^2}{Z(\omega, k)} J_1(kr)\exp(iq_z z - i\omega t)\,dk, \tag{1.1}$$

where $k_z = \sqrt{\omega^2/c_p^2 - k^2}$; $q_z = \sqrt{\omega^2/c_s^2 - k^2}$; $Z(\omega, k) = (2k^2 - \omega^2/c_s^2)^2 + 4k^2 q_z k_z$; $\mathbf{e}_\varphi$ is the unit vector in the cylindrical coordinate system $[r, \varphi, z)$; $\mu = \rho_0 c_s^2$ is the Lamé constant, and $J_n(x)$ is a Bessel function of the first kind. In the near zone, it is convenient to expand the field and its rotational components in powers of the small parameter $r^* = k_s r = r\omega/c_s$. Although in principle this expansion can also be obtained for z, the complexity of the results leads us to examine the limiting case $z \to 0$. For example, after obvious changes in variables, the expression for U_z becomes [8]

$$U_z = -\frac{\omega F_z}{2\pi\mu c_s}\int_0^{\infty+i\delta} \frac{J_0(pr^*)\, p\sqrt{p^2-\gamma^2}\,dp}{(2p^2-1)^2 - 4p^2\sqrt{p^2-\gamma^2}\sqrt{p^2-1}}, \quad \delta > 0.$$

Here integration along the real axis is shifted toward positive Im(p) to avoid the singular points of the integrand (1.1) in the proper manner.

The main component of the integral itself for $r^* \ll 1$ has large values of p on the order of $1/r^*$, and the magnitude of the component is of the same order. This means that the expansion starts with the term $1/r^*$. The corresponding coefficient can be found by integrating the asymptotic integrand for large p:

$$\frac{1}{2(1-\gamma^2)}\int_0^\infty J_0(pr^*)\,dp = \frac{1}{2r^*(1-\gamma^2)}.$$

Subtracting the asymptote for large p from the exact expression for the integrand and extrapolating r^* to zero yields the next term in the expansion. Continuing this method yields a series for u_z:

$$u_z = \frac{F_z}{4\pi\mu r(1-\gamma^2)}\left[1 - iIr^* - \frac{3-4\gamma^2+3\gamma^4}{4(1-\gamma^2)}(r^*)^2 + \ldots\right],$$

$$I = \int_0^\infty \frac{2(1-\gamma^2)s\sqrt{s^2+\gamma^2} + 4s^2\sqrt{s^2+1}\sqrt{s^2+\gamma^2} - (2s^2+1)^2}{(2s^2+1)^2 - 4s^2\sqrt{s^2+1}\sqrt{s^2+\gamma^2}}\,ds. \tag{1.2}$$

Here integration is carried out on the real axis, where the integrand has no singular points. The integral I is of order one and has a weak dependence on γ.

Calculation of the zero term in the expansion of $\boldsymbol{\psi}$ shows that for $z = 0$ the corresponding integral diverges at the upper bound, because the cross product of the field is not defined at the upper bound. The divergence vanishes for any finite $z > 0$, and the dependence on z is weak. Taking the limit of the series as $z \to 0$ yields the expression

$$\boldsymbol{\psi} = \frac{\mathbf{e}_\varphi F_z}{4\pi\mu r^2(1-\gamma^2)}\left[1 + \frac{3-4\gamma^2+3\gamma^4}{4(1-\gamma^2)}(r^*)^2 + \ldots\right]. \tag{1.3}$$

As expected, in the near zone the amplitude of $u_z(\sim r^{-1})$ varies an order of magnitude more slowly than the amplitude of $\boldsymbol{\psi}\,(\sim r^{-2})$. The zero term of the expansion is the limiting case of a stationary force $\mathbf{F}(\omega = 0)$ [7], and corrections to the amplitudes of $\boldsymbol{\psi}$ and u_z can be in-

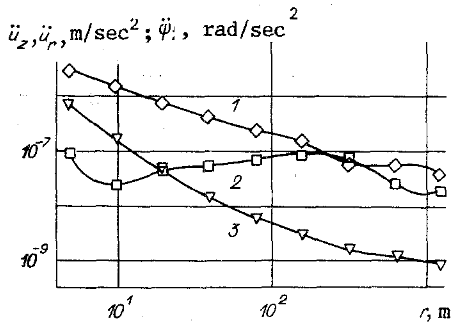

Fig. 1

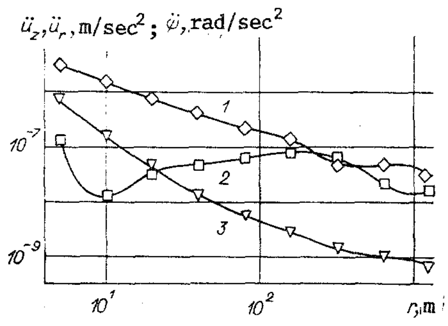

Fig. 2

tegrated for small values of r*, because the first term in the expansion of u_z is a phase correction.

Beyond the region where r* can be considered small and where Eqs. (1.2) and (1.3) apply, the components of the elastic field must be investigated by numerical methods. Here the model parameters were chosen to match a real experiment: c_s = 300 m/sec, z = 0.3 m. Figures 1 and 2 show calculated amplitudes $\ddot{u}_z$, $\ddot{u}_r$, and $\ddot{\psi}$ (curves 1-3) as a function of the distance r for a frequency of 1 Hz, c_p = 1600 m/sec (Fig. 1) or c_p = 2000 m/sec (Fig. 2), and $F_z = 10^4$ N. The asymptotic behavior of $\ddot{u}_z$ and $\ddot{\psi}$ as r → 0 agrees with the analytical results. Here no significant peculiarities in the behavior of the components were observed in the transition zone between the near and far zones. Figure 3 shows the phase difference Φ between value $\ddot{\psi}$ and $\ddot{u}_z$ which correspond to Figs. 1 and 2. The dependence of the results on c_p, that is, on γ, is insignificant.

2. We now examine a rigid sphere of mass m and radius a, which is embedded in an elastic material with parameters c_p, c_s, and ρ_0 [6]. The sphere is dynamically unsymmetric, with a geometric center at the point $\mathbf{r} = 0$ and a center of mass at $\mathbf{r} = \mathbf{r}_0$. The vector $\mathbf{r}_0$ lies in the plane y = 0, and the angle between the axis Oz and $\mathbf{r}_0$ is α_0. The sphere is excited by a plane longitudinal wave $\mathbf{U}_p = \mathbf{e}_z S_1 \exp(ik_p z - i\omega t)$ ($k_p = \omega/c_p$, $\mathbf{e}_z$ is the unit vector). The total displacement $\mathbf{U}$ of the elastic material satisfies the wave equation $\omega^2\mathbf{U} + c_p^2$ grad·div $\mathbf{U} - c_s^2$ rot rot $\mathbf{U} = 0$ and the boundary condition $\mathbf{U} = \mathbf{U}_0$ for $|\mathbf{r}| = a$ ($\mathbf{U}_0$ is the displacement vector for points on the surface of the sphere). The solution method in [6] can be used to provide an exact solution for both the translational and rotational movement of the spherical inclusion. Because [6] contains an error in the intermediate terms, we present the correct result for angular oscillation amplitudes of interest to us in the case where the inertial tensor of the sphere is diagonal. Rotation occurs around the Oy axis, and the corresponding rotation angle is

$$\psi_y = -\varepsilon q \cos\alpha_0 \frac{S_1}{a}\left[\left(\frac{I_y}{ma^2} - 2\frac{k}{\varkappa K^2}\right)(p-1) + \varepsilon^2 p\right]^{-1}. \tag{2.1}$$

where $\varepsilon = r_0/a$ is the relative eccentricity, I_y is the moment of inertia, $\varkappa = \rho_1/\rho_0$; ρ_1 is the density of the sphere, $K = k_s a$, $k_s = \omega/c_s$, $q = 9ih_2^{(1)}(K)/(\gamma^3 K^3 \Delta\varkappa)$, $p = [3h_2^{(1)}(K)\,h_2^{(1)}(\gamma K) + \Delta]/(\varkappa\Delta)$, $k = Kh_2^{(1)}(K)/h_1^{(1)}(K)$, $\gamma = c_s/c_p$, $\Delta = h_0^{(1)}(\gamma K)\,h_2^{(1)}(K) + 2h_2^{(1)}(\gamma K)h_0^{(1)}(K)$, and $h_n^{(1)}(x)$ is a spherical Bessel function of the third kind.

The case where the sphere is excited by a plane transverse wave $\mathbf{U}_s = \mathbf{e}_x S_2 \exp(ik_s z - i\omega t)$, is treated analogously. The single nonzero projection of the rotation angel ψ_y is

$$\psi_y = \frac{S_2}{a}(v\varepsilon\sin\alpha_0 - (p-1)\,w)\left[\left(\frac{I_y}{ma^2} - \frac{2k}{\varkappa K^2}\right)(p-1) + \varepsilon^2 p\right]^{-1}, \tag{2.2}$$

where $w = 3\left(\varkappa K^3 h_1^{(1)}(K)\right)^{-1}$; $v = 9ih_2^{(1)}(\gamma K)\,(K^3\Delta\varkappa)^{-1}$.

3. The correction to Eq. (0.1) in the long-wavelength region can be obtained from the expansion of Eqs. (2.1) and (2.2) for small K:

$$\psi_y = \frac{S_1}{a}\left[-\varepsilon\frac{\varkappa}{6}\cos\alpha_0 K^2\right] + O(K^4),$$

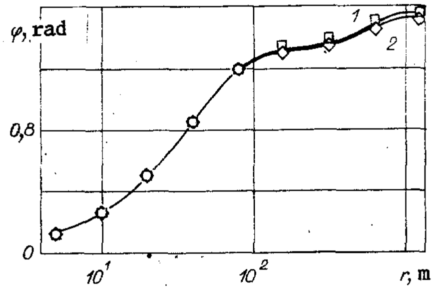

Fig. 3

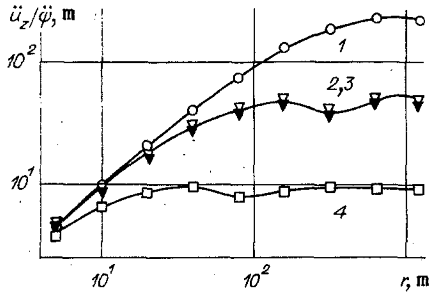

Fig. 4

$$\psi_y = \frac{S_2}{a}\frac{iK}{2}\left[1 - i\varepsilon\frac{\varkappa}{3}\sin\alpha_0 K + \frac{1}{6}\left(\varkappa\frac{I_y}{ma^2} + \varkappa\varepsilon^2 - 1\right)K^2\right] + O(K^4).$$

The form of this expansion shows that the second-order corrections to K are proportional to $\varkappa\varepsilon$, and their dependence on the angle α_0 of the incoming wave is different for different wave types. Therefore decreasing the mass of the detector can decrease the corrections.

We will show that measurements of the amplitudes of translational and rotational waves at a single point can be used to determine the direction and distance to a nearby surface source. Figure 4 shows the ratio of $\ddot{u}_z$ to $\Omega = \ddot{\psi}$ for various frequencies and elastic constants for c_s = 300 m/sec (curve 1: f = 0.2 Hz, c_p = 1600 m/sec; curve 2: f = 1 Hz, c_p = 1600 m/sec; curve 3: f = 1 Hz, c_p = 2000 m/sec; curve 4: f = 5 Hz, c_p = 1600 m/sec) as a function of r. On the linear section, the acceleration ratio is exactly equal to r and is independent of the elastic properties of the material, the frequency, and the oscillation source intensity; this independence is most important for practical applications. The distance can also be determined from the nonlinear sections by using calibration or the proper calculated corrections. Naturally, in this case the corrections will depend on a series of parameters, mainly on the wavelength $\lambda_s = 2\pi/k_s$. The direction to the source is determined easily from the orthogonality of the vectors $\ddot{\psi}$ and r ($\ddot{\psi}\cdot$r = 0). The components of the radius vector to the source can be computed from the vector product $[\ddot{u}_z \times \ddot{\psi}]$.

As a result, we conclude that along with the obvious appliction of translational-oscillation seismic detectors for differentiating wave types, these detectors also can be used to study the gradient characteristics of elastic fields, particularly in the near zone of an ocillating source.

LITERATURE CITED

1. B. B. Golitsyn, Selected Works [in Russian], Izd. Akad. Nauk SSSR, Moscow (1960).
2. V. A. Tokmakov and B. I. Kharin, "On the measurement of rotational oscillations with seismic instruments," Seismich. Prib., No. 12 (1979).
3. N. S. Lidorenko (ed.), Introduction to Molecular Electronics [in Russian], Énergoatomizdat, Moscow (1984).
4. L. I. Sedov, Mechanics of Continuous Media [in Russian], Nauka, Moscow (1987).
5. A. N. Guz', V. D. Kubenko, and M. A. Cherevko, Diffraction of Elastic Waves [in Russian], Naukova Dumka, Kiev (1978).
6. A. N. Kovshov and I. V. Simonov, "Several motions of a rigid sphere embedded in an unbounded elastic medium," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 5 (1967).
7. L. D. Landau and E. M. Lifshits, Theory of Elasticity [in Russian], Nauka, Moscow (1987).
8. V. T. Grinchenko and V. V. Meleshko, Harmonic Oscillations and Waves in Elastic Bodies [in Russian], Naukova Dumka, Kiev (1981).
9. V. S. Nikiforovskii and N. I. Onis'ko, "Investigation of the axisymmetric displacement field in an elastic half space," Fiz-Tekh. Probl. Razrab. Polezn. Iskop., No. 2 (1966).
10. V. N. Glaznev, V. S. Demin, and A. V. Solotchin, "Structure of a seismic field excited by a gas-dynamic source," in: Numerical Methods of Solving Problems of the Theory of Elasticity and Plasticity: Material from the Eighth All-Union Conference, V. M. Fomina (ed.), Novosibirsk (1984).